Expansion Theorem for Sturm-Liouville problems with transmission conditions

O. Sh. Mukhtarov[†] **and K. Aydemir**[†]

[†]Department of Mathematics, Faculty of Science,
Gaziosmanpaşa University, 60250 Tokat, Turkey
e-mail : omukhtarov@yahoo.com, kadriye.aydemir@gop.edu.tr

Abstract : The purpose of this paper is to extend some spectral properties of regular Sturm-Liouville problems to the special type discontinuous boundary-value problem, which consists of a Sturm-Liouville equation together with eigenparameter-dependent boundary conditions and two supplementary transmission conditions. We construct the resolvent operator and Green' s function and prove theorems about expansions in terms of eigenfunctions in modified Hilbert space $L_2[a, b]$.

Keywords : Boundary-value problems, transmission conditions, Resolvent operator, expansion theorem.

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1 Introduction

With historical roots in the application of Fourier series to heat flow, the Sturmian theory is one of the most extensively developing fields in pure and applied mathematics. As is well-known the eigenvalue parameter takes part linearly only in the differential equation in the classical Sturm-Liouville problems. However, in mathematical physics are encountered such problems, where eigenvalue parameter appear in both differential equation and boundary conditions. The first, we cite the works of Walter [18] and Fulton [7] both of which have extensive bibliographies, in the case of [7], a discussion of physical applications. Afterwards, we mention the results [3, 4, 11, 14] and corresponding references cited therein. In recent years there has been increasing interest of some Sturm-Liouville type problems which may have discontinuities in the solution or its derivative at interior point (see [1, 2, 5, 6, 8, 9, 10, 17, 19]).

In this paper we shall investigate some spectral properties of one discontinuous Sturm-Liouville problem for which the eigenvalue parameter takes part in both

differential equation and boundary conditions. Moreover, two supplementary transmission conditions at one interior point are added to boundary conditions. Namely, we consider the Sturm-Liouville equation,

$$\tau u := -u''(x) + q(x)u(x) = \lambda u(x) \tag{1.1}$$

to hold in finite interval (a, b) except at one inner point $c \in (a, b)$, where discontinuity in u and u' are prescribed by transmission conditions

$$\gamma_1 u(c-0) - \delta_1 u(c+0) = 0, \tag{1.2}$$

$$\gamma_2 u'(c-0) - \delta_2 u('c+0) = 0, \tag{1.3}$$

together with the eigenparameter- dependent boundary conditions

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0, \tag{1.4}$$

$$(\beta_1'\lambda + \beta_1)u(b) - (\beta_2'\lambda + \beta_2)u'(b) = 0, \tag{1.5}$$

where the potential $q(x)$ is real-valued, continuous in each interval $[a, c)$ and $(c, b]$ and has a finite limits $q(c \mp 0)$; $\alpha_i,\ \beta_i,\ \beta_i',\ \delta_i,\ \gamma_i\ (i = 1, 2)$ are real numbers; λ is a complex eigenparameter. Naturally we exclude each of the trivial conditions $\gamma_1 = \delta_1 = 0,\ \gamma_2 = \delta_2 = 0,\ \alpha_1 = \alpha_2 = 0,\ \beta_1' = \beta_1 = \beta_2' = \beta_2 = 0$. In contrast to previous works, eigenfunctions of this problem may have discontinuity at the one inner point of the considered interval.

This kind of problems are connected with discontinuous material properties, such as heat and mass transfer, varied assortment of physical transfer problems, vibrating string problems when the string loaded additionally with point masses, and diffraction problems [10, 16]. The study of the structure of the solution of the matching region leads to the consideration of an eigenvalue problem for a second order differential operator with piecewise continuous coefficients and transmission conditions at interior points. A. Boumenir [5] use sampling techniques to reconstruct the characteristic function associated with the eigenvalues of two linked SturmLiouville operators by a transmission condition. In [19], Wang et al. studied a class of Sturm-Liouville problems with eigenparameter-dependent boundary conditions and transmission conditions at an interior point. B. Chanane [6] computed the eigenvalues of SturmLiouville problems with several discontinuity conditions inside a finite interval and parameter dependent boundary conditions using the regularized sampling method. In [2] E. Bairamov and E. Uğurlu examined the determinants of dissipative Sturm-Liouville operators with transmission conditions. J. Ao et al. [1] have considered the finite spectrum of SturmLiouville problems with transmission conditions. Such properties as isomorphism, coerciveness with respect to the spectral parameter, completeness of

root functions, distributions of eigenvalues of some discontinuous boundary value problems with transmission conditions and its applications to the corresponding initial-boundary value problems for parabolic equations have been investigated in [8, 9, 10] and [16].

2 The fundamental solutions and Green' s function

By following the procedure of [9] we can define four solution $\phi_1(x,\lambda)$, $\phi_2(x,\lambda)$, $\chi_1(x,\lambda)$ and $\chi_2(x,\lambda)$ of the equation (1.1) under the initial conditions

$$u(a)=\alpha_2, \quad u'(a)=-\alpha_1, \tag{2.1}$$

$$u(c+0)=\frac{\gamma_1}{\delta_1}\phi_1(c-0,\lambda), \quad u'(c+0)=\frac{\gamma_2}{\delta_2}\frac{\partial\phi_1(c-0,\lambda)}{\partial x} \tag{2.2}$$

$$u(b)=\beta_2'\lambda+\beta_2, \quad u'(b)=\beta_1'\lambda+\beta_1, \tag{2.3}$$

and

$$u(c-0)=\frac{\delta_1}{\gamma_1}\chi_2(c+0,\lambda), \quad u'(c-0)=\frac{\delta_2}{\gamma_2}\frac{\partial\chi_2(c+0,\lambda)}{\partial x} \tag{2.4}$$

respectively. Consequently, each of the functions

$$\phi(x,\lambda)=\begin{cases}\phi_1(x,\lambda) \text{ for } x\in[a,c)\\ \phi_2(x,\lambda) \text{ for } x\in(c,b]\end{cases} \quad \chi(x,\lambda)=\begin{cases}\chi_1(x,\lambda) \text{ for } x\in[a,c)\\ \chi_2(x,\lambda) \text{ for } x\in(c,b]\end{cases}$$

satisfies the equation (1.1) and the both transmission conditions (1.2) and (1.3). Moreover, the solution $\phi(x,\lambda)$ satisfies the first of boundary condition (1.4), but $\chi(x,\lambda)$ satisfies the other boundary condition (1.5). By applying the same method as in [17] we can prove that the solutions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ are entire functions of complex parameter λ for each fixed $x\in[a,c)\cup(c,b]$.

It is known from ordinary linear differential equation theory that each of the Wronskians $w_1(\lambda)=W(\phi_1(x,\lambda),\chi_1(x,\lambda))$ and $w_2(\lambda)=W(\phi_2(x,\lambda),\chi_2(x,\lambda))$ are independent of x in $[a,c)$ and $(c,b]$ respectively. By using (2.2) and (2.4) we have

$$\begin{aligned} w_1(\lambda) &= \phi_1(c-0,\lambda)\frac{\partial\chi_1(c-0,\lambda)}{\partial x}-\chi_1(c-0,\lambda)\frac{\partial\phi_1(c-0,\lambda)}{\partial x} \\ &= \frac{\delta_1\delta_2}{\gamma_1\gamma_2}(\phi_2(c+0,\lambda)\frac{\partial\chi_2(c+0,\lambda)}{\partial x}-\chi_2(c+0,\lambda)\frac{\partial\phi_2(c+0,\lambda)}{\partial x}) \\ &= \frac{\delta_1\delta_2}{\gamma_1\gamma_2}w_2(\lambda) \end{aligned} \tag{2.5}$$

Denote

$$w(\lambda) := \gamma_1\gamma_2 w_1(\lambda) = \delta_1\delta_2\ w_2(\lambda). \tag{2.6}$$

Again, similarly to [9] it can be proven that, there are infinitely many eigenvalues $\lambda_n,\ n = 1, 2, ...$ of the BVTP (1.1) − (1.5) which are coincide with the zeros of characteristic function $w(\lambda)$.
Let us consider the nonhomegeneous differential equation

$$u'' + (\lambda - q(x))u = F_1(x), \tag{2.7}$$

on $[a, c) \cup (c, b]$ subject to nonhomogeneous boundary conditions

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0, \tag{2.8}$$

$$(\beta_1 u(b) - \beta_2 u'(b)) + \lambda(\beta_1' u(b) - \beta_2' u'(b)) = F_2 \tag{2.9}$$

and homogeneous transmission conditions

$$\gamma_1 u(c-0) - \delta_1 u(c+0) = 0, \tag{2.10}$$

$$\gamma_2 u'(c-0) - \delta_2 u'(c+0) = 0. \tag{2.11}$$

and let λ is not eigenvalue. Making use of the definitions of the functions ϕ_i, χ_i (i = 1, 2) we find that the general solution of the differential equation (2.7) can be written in the form

$$u(x,\lambda) = \begin{cases} \frac{\chi_1(x,\lambda)}{\omega_1(\lambda)} \int_a^x \phi_1(y,\lambda)F_1(y)dy + \frac{\phi_1(x,\lambda)}{\omega_1(\lambda)} \int_x^c \chi_1(y,\lambda)F_1(y)dy \\ \quad +c_{11}\phi_1(x,\lambda) + c_{12}\chi_1(x,\lambda)\ , \qquad for\ x \in (a,c) \\ \\ \frac{\chi_2(x,\lambda)}{\omega_2(\lambda)} \int_c^x \phi_2(y,\lambda)F_1(y)dy + \frac{\phi_2(x,\lambda)}{\omega_2(\lambda)} \int_x^b \chi_2(y,\lambda)F_1(y)dy \\ \quad +c_{21}\phi_2(x,\lambda) + c_{22}\chi_2(x,\lambda)\ , \qquad for\ x \in (c,b) \end{cases} \tag{2.12}$$

where C_{ij} $(i, j = 1, 2)$ are arbitrary constants. By substitution into the boundary conditions (2.8) and (2.9) we see at once that

$$c_{12} = 0,\ C_{21} = \frac{F_2}{\omega_2(\lambda)}. \tag{2.13}$$

Further, substitution (2.12) into transmission conditions (2.10) and (2.11)we have the inhomogeneous linear system of equations for c_{11} and c_{22} , the determinant of which is equal to $-\omega(\lambda)$ therefore is not vanish by assumption. Solving that system we find

$$c_{11} = \frac{1}{\omega_2(\lambda)} \int_c^b \chi_2(y,\lambda)F_1(y)dy + \frac{F_2}{\omega_2(\lambda)}, \tag{2.14}$$

$$c_{22} = \frac{1}{\omega_1(\lambda)} \int_a^c \phi_1(y,\lambda) F_1(y) dy. \tag{2.15}$$

Putting (2.13), (2.14) and (2.15) in (2.12) we deduce that problem (2.7)-(2.11) has an unique solution,

$$u(x,\lambda) = \begin{cases} \frac{\chi_1(x,\lambda)}{\omega_1(\lambda)} \int_a^x \phi_1(y,\lambda) F_1(y) dy + \frac{\phi_1(x,\lambda)}{\omega_1(\lambda)} (\int_x^c \chi_1(y,\lambda) F_1(y) dy \\ + \frac{\delta_1\delta_2}{\gamma_1\gamma_2} \int_c^b \chi_2(y,\lambda) F_1(y) dy + \frac{\delta_1\delta_2}{\gamma_1\gamma_2} F_2) \qquad for\ x \in (a,c) \\ \\ \frac{\chi_2(x,\lambda)}{\omega_2(\lambda)} (\frac{\gamma_1\gamma_2}{\delta_1\delta_2} \int_a^c \phi_1(y,\lambda) F_1(y) dy + \int_c^x \phi_2(y,\lambda) F_1(y) dy) \\ + \frac{\phi_2(x,\lambda)}{\omega_2(\lambda)} (\int_x^b \chi_2(y,\lambda) F_1(y) dy + F_2) \qquad for\ x \in (c,b) \end{cases} \tag{2.16}$$

By defining the Green' s function as

$$G_1(x,y;\lambda) = \begin{cases} \frac{\phi(x,\lambda)\chi(y,\lambda)}{\omega(\lambda)} & \text{for } a \le y \le x \le b, \quad x,y \ne c \\ \\ \frac{\phi(y,\lambda)\chi(x,\lambda)}{\omega(\lambda)} & \text{for } a \le x \le y \le b, \quad x,y \ne c \end{cases} \tag{2.17}$$

the formula (2.16) can be rewritten in the following form

$$\begin{aligned} u(x,\lambda) &= \gamma_1\gamma_2 \int_a^c G_1(x,y;\lambda) F_1(y) dy + \delta_1\delta_2 \int_c^b G_1(x,y;\lambda) F_1(y) dy \\ &+ \delta_1\delta_2 F_2 \frac{\phi(x,\lambda)}{\omega(\lambda)}. \end{aligned} \tag{2.18}$$

3 Operator formulation in modified Hilbert space

In this section we shall introduce a special equivalent inner product in the Hilbert space $L_2[a,b] \oplus \mathbb{C}$ and define a symmetric operator A in this space such a way that the considered problem can be interpreted as the eigenvalue problem of this operator. For this we assume that, $\rho := \beta_1'\beta_2 - \beta_1\beta_2' > 0$ and for the sake of shorting we restrict ourselves to the investigation only the case $\gamma_i \ne 0, \delta_i \ne 0 (i = 1,2)$.
In the Hilbert Space $H = L_2[a,b] \oplus \mathbb{C}$ of two-component vectors we define an equivalent inner product by

$$< F,G >_1 := |\gamma_1\gamma_2| \int_a^c F_1(x)\overline{G_1(x)} dx + |\delta_1\delta_2| \int_c^b F_1(x)\overline{G_1(x)} dx + \frac{|\delta_1\delta_2|}{\rho} F_2\overline{G_2}$$

for $F = \left(\ F_1(x), F_2 \ \right), \quad G = \left(\ G_1(x), G_2 \ \right) \in H$ and apply operator theory in the modified Hilbert space $H_1 = (L_2[a,b] \oplus \mathbb{C}, <.,.>_1)$. Below we shall use the following notations:

$$(u)_\beta := \beta_1 u(b) - \beta_2 u'(b), \quad (u)'_\beta := \beta'_1 u(b) - \beta'_2 u'(b).$$

Let us define a linear operator $H : A \to A$ with the domain

$$D(A) := \Big\{ \ F = \left(\ F_1(x), (F_1)'_\beta \ \right) : F_1(x) \text{ and } F'_1(x) \text{ are absolutely continuous in each interval [a,c) and (c,b], and has a finite limits } F_1(c \mp 0) \text{ and } F'_1(c \mp 0), \ \tau F_1 \in L_2[a,b], \ a_1 u(a) + a_2 u'(a) = 0, \ \gamma_1 F_1(c-0) = \delta_1 F_1(c+0), \ \gamma_2 F'_1(c-0) = \delta_2 F'_1(c+0) \Big\}$$

and action low

$$A(F_1(x), (F_1)'_\beta) = (\tau F_1, (-F_1)_\beta).$$

Consequently the problem $(1.1) - (1.5)$ can be written in the operator form as

$$AU = \lambda U, \quad U := (u(x), (u)'_\beta) \in D(A)$$

in the Hilbert space H_1.

Lemma 3.1. *The domain $D(A)$ is dense in H_1.*

Proof. □

Theorem 3.2. *If $\gamma_1\gamma_2\delta_1\delta_2 > 0$ then the linear operator A is symmetric.*

Proof. □

Remark 3.3. *Having in view this property of the problem $(1.1) - (1.5)$, we shall assume $\gamma_1\gamma_2\delta_1\delta_2 > 0$ everywhere in below. Also without loss of generality we shall let $\gamma_1\gamma_2 > 0$ and $\delta_1\delta_2 > 0$.*

Remark 3.4. *By Lemma 3.2 all eigenvalues of the problem $(1.1) - (1.5)$ are real. Consequently, we can now assume that all eigenfunctions are real-valued.*

Corollary 3.5. *Let $u(x)$ and $v(x)$ be eigenfunctions corresponding to distinct eigenvalues. Then*

$$\gamma_1\gamma_2 \int_a^c u(x)v(x)dx + \delta_1\delta_2 \int_c^b u(x)v(x)dx + \frac{\delta_1\delta_2}{\rho}(u)'_\beta(v)'_\beta = 0. \tag{3.1}$$

Proof. The proof is immediate from the fact that, the eigenelements $(u(x), (u)'_\beta)$ and $(v(x), (v)'_\beta)$ of the symmetric linear operator A is ortogonal in the Hilbert space H_1. $\square$

4 The Resolvent operator and Self-adjointness of the problem

In this section we shall construct the Resolvent operator and prove self-adjointness of the problem.

Lemma 4.1. *Let λ_0 be zero of $w(\lambda)$. Then the solutions $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are linearly dependent.*

Proof. $\square$

Theorem 4.2. *Each eigenvalue of the problem (1.1)-(1.5) is the simple zero of $w(\lambda)$.*

Proof. $\square$

Let A be defined as above and let λ not be an eigvalue of this operator. For construction the resolvent operator $R(\lambda, A) := (\lambda - A)^{-1}$ we shall solve the operator equation

$$(\lambda - A)U = F \tag{4.1}$$

for $F = (F_1(x), F_2) \in H_1$. This operator equation equivalent to the problem (2.7)-(2.11).
Using the equalities we see that

$$(G_1(x, .; \lambda))'_\beta = \frac{\phi(x, \lambda)}{\omega(\lambda)} (\chi(x, \lambda))'_\beta = \rho \frac{\phi(x, \lambda)}{\omega(\lambda)}. \tag{4.2}$$

Hence, the solution (2.16) may be written as

$$\begin{aligned} u(x, \lambda) &= \gamma_1\gamma_2 \int_a^c G_1(x, y; \lambda) F_1(y) dy + \delta_1\delta_2 \int_c^b G_1(x, y; \lambda) F_1(y) dy \\ &+ \frac{\delta_1\delta_2}{\rho} (G_1(x, .; \lambda))'_\beta F_2 \end{aligned} \tag{4.3}$$

Consequently, for the solution

$$U(F, \lambda) := (u(x, \lambda), (u(., \lambda))'_\beta) \tag{4.4}$$

of the nonhomogeneous operator equation (4.1) we obtain the following formula

$$U(F,\lambda) := (< G_{x,\lambda}, \overline{F} >_1, (< G_{x,\lambda}, \overline{F} >_1)'_\beta) \tag{4.5}$$

where

$$G_{x,\lambda} := (G_1(x, .; \lambda), (G_1(x, .; \lambda))'_\beta) \tag{4.6}$$

Now, making use (2.17), (4.3), (4.4), (4.5) and (4.6) we see that if λ not an eigenvalue of A then

$$U(F,\lambda) \in D(A) \text{ for } \ F \in H_1, \tag{4.7}$$

$$U((\lambda - A)F, \lambda) = F, \text{ for} \in D(A) \tag{4.8}$$

and

$$\|U(F,\lambda)\| \leq \ |Im\lambda|^{-1}\|F\| \text{ for } F \in H_1, \quad Im\lambda \neq 0. \tag{4.9}$$

Hence, each nonreal $\lambda \in \mathbb{C}$ is a regular point of an operator A and

$$R(\lambda, A)F = (< G_{x,\lambda}, \overline{F} >_1, (< G_{x,\lambda}, \overline{F} >_1)'_\beta) \text{ for } \ F \in H_1 \tag{4.10}$$

Because of (4.7) and (4.10)

$$(\lambda - A)D(A) = (\overline{\lambda} - A)D(A) = H_1 \text{ for } Im\lambda \neq 0. \tag{4.11}$$

Theorem 4.3. *The linear operator A is self-adjoint.*

Proof. From the equality (4.11) and the fact that A is symmetric it follows by the standard theorems for symmetric operators in Hilbert spaces that A is self-adjoint in H_1 (see, for example, [12], Theorem 2.2.p. 198). □

5 Expansion is series of eigenfunctions

Let $\lambda_n,\ n = 1, 2, ...$ are eigenvalues of the operator A and let $\phi_n(x) := \phi(x, \lambda_n), n = 1, 2, ...$ be defined as in section 2. By virtue of Lemma 4.1 the two-component vectors

$$\Phi_n := (\phi(x, \lambda_n), (\phi(., \lambda_n))'_\beta), \quad n = 0, 1, 2, ... \tag{5.1}$$

are the eigenelements of A. Moreover,

$$< \Phi_n, \Phi_m >_1 = 0 \quad for\ n \neq m \tag{5.2}$$

since A is self-adjoint in H_1. Denote the normalized eigenelements by

$$\Psi_n := (\psi_n(x), (\psi_n(x))'_\beta), \tag{5.3}$$

where

$$\psi_n(x) := \frac{\phi(x,\lambda_n)}{\|\Phi_n\|_1}. \tag{5.4}$$

Let $k_n \neq 0$ denote the real constant for which

$$\chi(x,\lambda_n) = k_n \phi(x,\lambda_n), \quad n = 0,1,2,...\ x \in (a,c) \cup (c,b). \tag{5.5}$$

Then

$$(\phi_n(x))'_\beta = \frac{\rho}{k_n}. \tag{5.6}$$

Writing for λ_n instead of λ_0 we obtain

$$\|\phi_n\|_1^2 = \frac{\omega'(\lambda_n)}{k_n}. \tag{5.7}$$

Now, making use the representation (4.3) of the solution $u(x,\lambda)$, the equalities (2.17), (5.3)- (5.5) and the fact that each eigenvalue λ_n is simple zero of $\omega(\lambda)$ we derive that

$$Res_{\lambda=\lambda_n} u(x,\lambda) =< F, \Psi_n >_1 \psi_n(x). \tag{5.8}$$

Consequently,

$$Res_{\lambda=\lambda_n} R(\lambda, A)F :=< F, \Psi_n >_1 \Psi_n = C_n(F)\Psi, \tag{5.9}$$

where

$$C_n(F) :=< F, \Psi_n >_1 \tag{5.10}$$

are Fourier coefficients.

Theorem 5.1. *(i) The modified Parseval equality*

$$\begin{aligned} \gamma_1\gamma_2 \int_a^c f^2(x)dx + \delta_1\delta_2 \int_c^b f^2(x)dx &= \sum_{n=0}^{\infty} \mid \gamma_1\gamma_2 \int_a^c f(x)\psi_n(x)dx \\ &+ \delta_1\delta_2 \int_c^b f(x)\psi_n(x)dx \mid^2 \end{aligned} \tag{5.11}$$

is hold for each $f \in L_2[a,c] \oplus L_2[c,b]$.

Proof. □

Theorem 5.2. *Let* $(f(x), (f)'_\beta) \in D(A)$. *Then*

$$\begin{aligned}(i)\quad f(x) &= \sum_{n=0}^{\infty} \Big(\gamma_1\gamma_2 \int_a^c f(x)\psi_n(x)dx + \delta_1\delta_2 \int_c^b f(x)\psi_n(x)dx \\ &+ \frac{\delta_1\delta_2}{\rho}(f)'_\beta(\psi_n)'_\beta\Big)\psi_n(x) \qquad (5.12)\end{aligned}$$

where, the series converges absolutely and uniformly in whole $[a, c) \cup (c, b]$. *(ii) The series (5.12) may also be differentiated, the differentiated series also being absolutely and uniformly convergent in whole* $[a, c) \cup (c, b]$.

Proof. □

6 Counterexample

Recall that we had derived all results in this study under condition $\gamma_1\gamma_2\delta_1\delta_2 > 0$. Let us show that this simple condition on the sign of the coefficients of the transmission conditions can not be omitted without putting any other condition on this coefficients. For this, consider the following special case of the problem (1.1)-(1.5) for which $\gamma_1\gamma_2\delta_1\delta_2 < 0$:

$$-u'' = \lambda u, \qquad x \in [-1, 0) \cup (0, 1] \tag{6.1}$$

$$u(-1) = 0 \quad , \quad \lambda u(1) = u'(1) \tag{6.2}$$

$$u(0-0) = u(0+0) \quad , \quad u'(0-0) = -u'(0+0) \tag{6.3}$$

It is easy to verify that this problem has only the trivial solution $u = 0$ for any $\lambda \in \mathbb{C}$. Thus, if $\gamma_1\gamma_2\delta_1\delta_2 < 0$ then the spectrum of the problem (1.1)-(1.5) may be empty.

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